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A windows app that gamifies the process of learning piano chords by listening to the player

H446

A-Level Computer Science

A-Level Computer Science

Practical Programming Project

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Analysis

# Summary

I would like to create a windows app that gives a piano chord then listens for the chord to be played, then automatically gives a new chord.

## Further Problem Identification

Currently there is not a good app to help you practice playing chords on the piano. It is vital to practice playing chords to become a better player, but it can become boring and repetitive. Therefore, I want to create an app to gamify the experience of learning new chords by timing how long it takes you to play the chord. This app could also keep track of what chords you are competent in, and show them less, and what chords you are not competent in playing and show them more. This competency rating can be calculated from the time that it takes you to play the chord. Through this, it can help the user to learn the chords through a fun game-like approach rather than boring repetition and practice. This will increase user satisfaction with piano learning, making it a more satisfying and interesting experience.

## Computation

The solution is applicable for computational methods for many reasons. Firstly, the process of listening for a sound and comparing it to a known sound or note can be done quite easily by inputting a section of sound and doing a Fourier transform to convert the wave to a list of frequencies present. I can then discard all the frequencies whose volumes are below a certain threshold, and then convert the remaining frequencies to notes. The computer can automatically display a new chord if the correct notes are detected, whilst keeping track of all the previous chords played and how quickly, to train the user to play the chords much quicker.

### Decomposition

The project can be split into three main parts which should be able to work independently of each other:

* Microphone/Listener/Note Identifier
* UI
* Game – question storage and tracker

By splitting the project in this way, I can ensure that the project can be adapted to meet the needs of any other potential clients other than the stakeholders, so that the code is versatile, modular and works on many devices. I should also be able to employ abstraction in these three areas, so that when developing the UI, I do not need to code the game directly, I only need to interface with it. Furthermore, it will make the project easier to debug because I will be able to more easily identify in which section of the project the bug is in, rather than having to debug the entire code.

The solution involves an algorithm which has some steps:

1. Listen for a sound.
2. Convert the sound from a sine wave to a frequency chart by Fourier transform.
3. Scan the frequencies found to compare against expected frequencies of notes.

This is how the computer will detect chords that the user will play. To single out any notes from background noise, I will only accept the frequencies whose amplitudes are above a certain threshold.

### Divide and Conquer

Solving the above decomposed problems together seems computationally intensive. To be able to write my solution to the problem efficiently and easily, I will need to conquer each of the decomposed problems separately. I will even divide these components into smaller algorithms and subprograms that seem more manageable on their own. For example, I will first handle the background tasks such as microphone to notes before I do the game and then the UI.

Each component in the abstraction section can be programmed separately so that they fit together modularly. This will make it easier to expand on the app in the future, and to build each section of the app.

### Abstraction

Each part of the solution will be abstracted from the others. At the top, there will be the UI, and what the user sees. Then, underneath that there will be the game, which will load new chords and keep track of the user’s competency of the chords. Underneath the game, there will be two components: the file system and loading, because it would be a good idea to store the chords in secondary storage so that the user can save their progress. Also, there will be the note identifier, which takes in frequencies and compares it to notes then outputs any notes that it hears. There will be one more component underneath that which listens and converts the sine wave to frequencies.

My reasons for splitting it this way is that these “modules” are separate to each other and although they need each other to run, they do not need to know what the other modules do. I can program them completely separately, so that I can modify one of the modules and it won’t affect how the other modules run. Furthermore, if I wanted to add more functionality in the future, I would just add more modules the current tree, and I would not need to change much of the original modules to do that.

### Data Mining

The project will implement a simplified version of data mining where I will collect data such as time to find each chord, how many wrong notes were played before the chord was detected, etc. I can then use this data to show the user chords that they find more difficult to play more often, so that they will learn faster.

### Threading

To solve my problem, I will need to utilise multiple threads, as the computer will need to be listening, counting, and checking the previous notes at the same time. My threads will need to pass data to one another so I will implement a lock on some global variables to allow this. One example of where this will be useful is the thread that is listening passes the frequencies to the game, to check that the notes are correct.

### Conclusion

Because of all the above-mentioned computational methods, the solution is very clearly solvable by computational methods. In fact, the solution can only be solved effectively by computational methods because a human would not be able to identify the chords or provide new ones with enough accuracy.

# Stakeholders

# Interviews

**Key questions for James & Sarah:**

1. Are you satisfied with how you currently learn to play piano?
2. Have you ever tried an app to help you learn, and were you satisfied with how the app helped?
3. Do you often play by chord patterns or do you play by reading the music in its entirety.
4. Do you feel like an app to help you practice a specific area of piano would be better than a general-purpose app?
5. Would you prefer more of a game-y app or a revision app?
6. Is there anything that you would love to see in an app like this?

Questions 1 and 2 establish whether they are happy with how they learn. This is important because it depends on this how they use the app, and whether it completely re-shapes their learning or just is a fun game.

Questions 3 and 4 enquires about a need for a specific app that is for more advanced players, rather than a “learn piano from scratch” kind of app.

Question 5 helps me to understand how game-y the app should be. This is important because some apps go to far, and some don’t go far enough. For example, you don’t want to be running around a 3D world whilst playing your chords, it’s too far, but also, you don’t want to be staring at a grey screen telling you chords to play with no reward or game at all.

**Key questions for Mr Johnson:**

1. Are you satisfied with how your tutees currently learn to play piano?
2. Have you ever tried an app to help you teach, and were you satisfied with how the app helped?
3. Do you often teach to play by chord patterns or do you prefer playing by reading the music in its entirety.
4. Do you feel like an app to help teach a specific area of piano play would be better than a general-purpose app?
5. Would you prefer your students to learn from more of a game-y app or a revision app?
6. Is there anything that you would love to see in an app like this?

Questions 1 and 2 establish whether they are happy with the resources available for teaching. This is important because it depends on this how they use the app to help their students, and whether it completely re-shapes their teaching or just is a fun game.

Questions 3 and 4 enquires about a need for a specific app that is for more advanced players, rather than a “learn piano from scratch” kind of app.

Question 5 again helps me to understand how game-y the app should be.

**James**

1. No because at the moment I do not have time to practice as much as I would like, however I feel like when I do have the time, I can improve quickly in pieces I’m learning. I do find that my lessons with my teacher every Friday are very valuable to me though.
2. No I haven’t because my teacher is very good and trained me in classical piano and therefore, I do not feel that an app would improve my ability.
3. I find hat reading music in its entirety is the only way to play classical piano, chord patterns are more useful in Jazz/blues music.
4. I would prefer a general app that can be used to also practice specific areas such as scales, sight reading, understanding music etc.
5. Revision because I am more bothered about targeted focus and serious practice.
6. Sight reading trainer because being able to sight read well means learning the notes to new pieces becomes much easier although of course the finer details can take just as long to master.

**Sarah Smith**

1. No, not really. I can’t play my favourite songs, and it’s taking too long to learn the chords. I use online lessons to try and improve but they’re quite boring.
2. I watch YouTube tutorials to learn piano because, being in college, I haven’t enough money for a private tutor or a subscription-based app.
3. I do a bit of both really. If the chords on the music look a bit hard, I’ll figure them out from the letters, and if I don’t know what the letters mean, I’ll usually work it out from the music or look it up online.
4. Definitely! I feel like the general-purpose apps don’t tailor to the needs of the play at all. I’m quite good at timing and melody so I don’t need to practice that as much.
5. I’m a huge fan of anything that makes learning more fun, so definitely a more game-y app.
6. I’d love to see colour-blind support and easy to recognise visual representations of the chord patterns, because I’m colour-blind, and I have dyslexia.

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These stakeholders an interested in a game-revision hybrid, that is fun to learn, but also makes sure you do a suitable amount of learning. They are not interested in any other game components that wouldn’t contribute to their learning.

Sarah pointed out that there needs to be accessibility features such as colour-blind settings, and visual arrangements of the chords for those who have dyslexia.

Ian said that he didn’t have much time to learn, so the sessions need to be quick and concise, with no ads or annoying pop-ups.

**Mr Johnson**

1. Mostly. My students sometimes get a bit bored memorising theory, and they just want to be playing songs, but unfortunately you need to know the chords to play the songs.
2. I’ve looked at some, but they’re only good for complete beginners, “find the note” type apps, if you know what I mean.
3. I teach both, as they are both important skills to have. Students should be able to read the music but also it can be quicker to read the chords if the player is competent enough. It entirely depends on which they are better at.
4. I hate the general-purpose apps, because they act like they can replace a tutor, but they don’t teach half the stuff like chords, posture, etc.
5. I think they’d love the game-y app, but they do need to learn, so it needs to be fun but also educational.
6. I would love to see the ability for me or the student to select chords they want to practice more that session, so that I can set them tasks to practice certain chords.

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Mr Johnson seemed to agree with the other stakeholders, that the app needs to be fun but educational. He also highlighted the need for there to be some control over the chords that are learnt in a session. Furthermore, he pointed out that the app should be a tool to help, not to completely overthrow teachers.

## Research

#### “Yousician”

A screenshot of a music website

Description automatically generated

* An app that helps people learn various instruments by showing the notes and listening to the user play them. Doesn’t focus on chords, but more like reading traditional sheet music.
* Things I like and could incorporate into my app:
  + Plays the chord so that you can learn to hear what it should sound like. This is needed so that the user can learn to listen to the chords, and this helps ear training.
  + Gamify, score system. This keeps the user engaged with the app for longer so that they can learn more.
  + Piano at bottom of screen showing notes. This helps beginners to find the notes quicker and see the chord patterns.
* Things I don’t like:
  + Too many tutorial videos. The user should spend most of their time practicing than watching videos, and the learning points should be developed into the game.
  + Tailored much more towards beginners. I need my app to be catered towards many difficulties to suit many types of piano players.
  + Too much background tune. This tune can be fun for the user but it’s harder to recognise the chords whilst learning and the melody should only be played when the song itself is in practice mode, rather than all the time.

#### “Simply Piano”

A screenshot of a video game

Description automatically generated

* A large, versatile piano app that helps the user to learn how to play popular songs. Overly broad range of things you can do on it, not great for more advanced learners.
* Provides songs based on the courses that the user has completed.
* Shows sheet music, and helps you learn all aspects of music from the very beginning.
* Some things I could take from this include:
  + The sheet music to show which notes to play. This can teach the user to read the music as well as play the chords.
  + Videos/cartoon graphics showing which notes to play. These should be short and concise to keep engagement high.
  + The learning curve and complexity scale. This is a good example of how to tailor an app to multiple difficulties and react to how easy the user finds the chords.
  + The UI is neat and minimal – quite simple and no way to get confused. This is good for users with a range of technological experience.
* Some things I don’t like about Simply Piano include:
  + Only available for mobile devices, which tend not to have as good a microphone as a computer with a stand-alone microphone. Also, mobile devices often do lots of hardware manipulation to the microphone to try and single out the voice for calls. This is less good if you want to record not a voice.
  + You can’t skip around the music or slow down and speed up the tempo.
  + Expensive subscription.
  + No option to turn of backing track. The backing track was annoying especially when trying to play your own thing.

### Conclusion

Based on this research, I can conclude that I do not want to over-complicate the app, as that is the downfall of many of the larger piano-learning apps. They try to listen to whole songs and rhythm too, and most of the time it doesn’t work very well. There are a few apps which attempt to do something similar, but they don’t often do exactly what my stakeholders want and are often bloated and do lots of things not very well instead of one thing very well.

## Essential features

### Display Chord

The chord name (e.g. Cmin11) should be displayed boldly as the main subject in the screen. The chord name should update on each round of the game when the user moves on to the next chord. This can either be by getting it right or getting it wrong.

### Listen to Notes

After a chord is displayed, the app needs to listen to the user playing notes to deduce if the user is playing the correct notes. This can be done by listening to small amounts of audio regularly through the microphone and doing a FFT to determine the frequencies present, and then checking the frequencies to make sure the notes are correct.

### Gather Data

Whilst the user is playing, the app should gather data such as how long it took the user to get the answer correct and how many tries it took, or incorrect notes. This is so that the app can get a good idea of which chords the user knows better than others and give these chords less.

### Learning Algorithm

The app should then use the data collected to give the chords to the user in a better order, giving some chords more frequently depending on the user’s apparent competence in the chord. This is justifiable because the clients said they need a way of tailoring the chords that are asked to the ability of the user.

### Sheet Music

As per my stakeholder’s request, there should be an option to play the chords by sheet music as well as chord name.

## Limitations

### Time

One of the biggest limitations is time. I have only a couple of months to design and create this app, so some features must be left out. However, in the section labelled “Optional Features”, there are features that could be implemented if I have time, features that would be nice to have but aren’t essential.

### Melody/Tempo

The app will not have capacity to recognise the tempo or rhythm of the user’s playing, just the notes that they play. This is because it is difficult to measure a melody because students play at a wide variety of speeds. Furthermore, chords don’t require rhythm or tempo to play when they are not in the song, so it is purely outside of the scope of the app, which doesn’t teach the user songs but the chords.

## Optional Features

### Song mode

The user could play along to a song that the app plays and displays the chords next to it. This could be a fun game for the user and the user could even learn the melody then not even need the chord.

### Piano Graphic

There could be a piano graphic at the bottom of the screen which highlights keys based on which ones the user should play kind of like a typing game.

## Requirements

The user’s device must have a microphone to detect the notes, as well as support for .exe programs, so it must be a windows device. Without a microphone, the app should “strongly suggest” plugging in a microphone, but still work, moving on with a button click instead. The user must have a piano or keyboard that is in tune with standard tuning.

### Testing and Development

Visual Studio C# Win Forms Application – a useful IDE to develop and test the code. “The Visual Studio IDE is a creative launching pad that you can use to edit, debug, and build code, and then publish an app. Over and above the standard editor and debugger that most IDEs provide, Visual Studio includes compilers, code completion tools, graphical designers, and many more features to enhance the software development process”[[1]](#footnote-1)

## Success Criteria

|  |  |  |  |
| --- | --- | --- | --- |
| Criteria Number | Description | Justification | Achieved? |
| 1 | Note Detection |  |  |
| 1.1 | The program listens and saves the recording in RAM | Needs to be able to listen to the user |  |
| 1.2 | The program converts the sine wave to frequencies | Needs to be able to interpret the sound |  |
| 1.3 | The frequencies are translated into notes | Needs to be able to figure out which notes are being played |  |
| 1.4 | The notes are saved | So that the program can read the notes and compare them to the notes it expects to hear |  |
| 2 | Difficulty/Revision – Choice of chords |  |  |
| 2.1 | The program stores lots of chords, and the chord’s notes in a large structure. | So that the chords are available quickly and an algorithm can figure out which chords to choose |  |
| 2.2 | The program chooses a chord based on some data collected | So that when the user starts getting it correct quickly, the chord plays less |  |
| 2.2.1 | Each chord has a “score” which is how well the user knows it |  |  |
| 2.2.2 | The score is re-calculated each round based on data |  |  |
| 2.3 | Chords are chosen by score, but never repeated directly (can be ABAB but not AABB) | So the user doesn’t get the same chord more than once in a row |  |
| 2.4 | The user can select a difficulty | So that advanced users don’t have to start on easy chords |  |
| 2.5 | The user can set favourite chords which have a much higher “score”, so they are played more | So that the piano player can select specific chords they want to practice, for example the tutor wants the student to practice the “Gmin” chord patterns |  |
| 3 | Game |  |  |
| 3.1 | The game gives the user a chord | So the user knows what to play |  |
| 3.2 | The game checks any notes currently being played against the chord notes | So the game knows if the notes are correct |  |
| 3.3 | If the notes are correct, the game gives the user a new chord | So the user knows if the notes are correct |  |
| 3.4 | There is a score that counts how many chords the user has correct, like a streak | This helps the game be more of a game and less a revision tool |  |
| 3.5 | The game shows the user the correct answer after it is played incorrectly | This helps the user to learn the chord |  |
| 3.6 | The user can favourite a chord mid-game | So that the user can go back and reflect upon the chords |  |
| 3.7 | The user can press “next” to skip a chord | So that if the user doesn’t know they are not stuck in a loop |  |
| 4 | UI |  |  |
| 4.1 | The GUI is clean and simple | So that people who might not know how to use computers competently can still use the app |  |
| 4.2 | The score is displayed | This is purely for gamification |  |
| 4.3 | The current chord is displayed | .. |  |
| 4.4 | There is indication that the chord is favourited, difficult, etc. | So the user knows whether they have favourited it or not |  |
| 4.5 | The background changes to green with ticks when the answer is correct, and red with crosses when the answer is incorrect. | This is because one of my stakeholders is colour blind, so they won’t know the difference between read and green. However, for those non-colourblind it still needs to be visually appealing and easy to understand briefly |  |
| 4.6 | 4.5 is easy to understand for a colour-blind person | .. |  |
| 4.7 | Next button/Help button | For the user to select the next chord just in case they don’t know the answer |  |
| 4.8 | Streak must be displayed | Motivation and gamification |  |
| 4.9 | Some indication of the correct chord visually | So the user can more easily find the chord on the piano |  |
| 5 | Catches/robustness |  |  |
| 5.1 | No microphone – recommends to the user that they should plug in a mic but continues anyway.  This should also trigger if there is little or no sound | This is so that the user can practice their recall of the chords like flashcards if they are out and about. |  |
| 5.2 | Badly Tuned piano – does its best but doesn’t respond to notes that are too far away from the accurate frequency | This is so that there is less chance of an error |  |
| 5.3 | All inputs validated | So there is little chance of crashing if the user wants chord number -1 to be displayed. |  |
| 5.4 | Notes that are too loud or quite should be dealt with  Use of a decibel scale could help here. | If the notes are too quiet on average, I still need them to be picked up by the microphone, and if there is some background noise then some loud notes, I need to differentiate between them so that the background noise isn’t perceived as notes. |  |

Design

# Decomposition

I am going to break the problem down into a series of smaller problems, which are individually suitable for computational solution. Below is a diagram showing the different modules that I am going to need to build.

At the top of the tree is the app, which splits into the “Main Game”, which contains all the background coordination and timing, and the “UI”. I have split it like this because the UI should not need to know anything about the game or background application and should work as on its own. Moving depth first, under the “Main Game”, there is a timer to help the game run and to track the user’s response time. There is also the part of the game which coordinates the rest of the background processes. I specified that this was part of the game because the game controls what happens and when, and is generally in charge of the runtime of the app. These background processes include the revision engine, which controls the progress and which chords are selected for the user. As per my stakeholder’s needs, the revision engine gives chords based on the chords that have previously been played, and the user’s confidence on them. Built into this is the favourite system, which one of my clients requested. Another main part of the background processes is the listener, which controls the microphone and input and converting that to notes. I have put this at the bottom of the tree because the game, UI, or anything else does not need to know how it works, only the output from it.

Under the UI section of the game, there is all the components that need to work together for the UI. The reason I have added the microphone into this is because I will need to plot microphone levels for debugging purposes. A scaled down version of this can be in the final app to help the user to debug their microphone.

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Description automatically generated

# Structure of the Solution

I am going to use a more agile method and therefore I will regularly check in with my stakeholders to get their feedback on the solution. This is so that I can ensure that I am making the app that they want.

For the solution, I am going to use and object-oriented approach. This will include at least the following classes:

## Listener

This class oversees the process of getting the notes from the sine wave at regular intervals. It contains some constants, including the rate, buffer size and device number. These constants make sure that all the data can be modified in the same way because the sound waves have the same properties. This class also has many methods:

**Listener**

This is the constructor for the algorithm, and it crucially allows the setting of the wave properties and constants in the attributes section. It also contains default values for the properties that match what is expected for the wave functions to be, e.g. 44100kHz, device number 1. This method will also create a buffer that the sound will be temporarily stored in whilst it is scanned.

**waveIn\_DataAvailable**

This is an event handler for if there is any data available from the microphone. It has the simple job of adding the samples to the buffer so that another process can read it.

**StartListening**

This is a method that starts the recording of audio from the mic. More specifically, it should initialise the buffer and set the waveIn device to recording.

**WhatNoteAmI**

This method works out, from the frequency that it is given, the note that is played. This is done by these equations:

This equation works out the MIDI number (fig. 1) by comparing it against the frequency of a known note, A, which has frequency 440Hz, and is MIDI number 69. We use log base 2 here because if you double the frequency of a note, you get the same note one octave higher (A5 would be 880Hz). This means that the scale for increase of pitch is logarithmic not linear. This is because our perception of pitch is also logarithmic.  
The MIDI Number should be rounded to a whole number here (Success Criteria 5.2)

To work out the actual note name from the MIDI number, we can just use a list of all the note names:

{ "A", "A#/Bb", "B", "C", "C#/Db", "D", "D#/Eb", "E", "F", "F#/Gb", "G", "G#/Ab" }

And go to the index worked out here:

We take 21 because the smallest MIDI number (A0) is is 21. Then modulo because there are 12 items in the list and I don’t care about the octave that is played.



**Process Data**

This method will regularly be called to process the most recent data in the microphone. It will call other methods to help it do this.

**Calculate PCM Values**

PCM (pulse-code modulation) is a way of digitally representing analogue signals. The amplitude of the signal is sampled at uniform intervals and quantized to the nearest value within a range of steps.[[2]](#footnote-2)

The method has an input of a byte array containing the information in the buffer, and how many points of data there are. The method should then output a double array of values. These values are the integer numbers stored in every 2 bytes of the array as a percentage of the maximum 16 bit integer.

This array is then the output.

**Calculate FFT Values**

Similarly to PCM values, this method takes an input of the byte array and the number of samples of data. It also takes an input of the PCM array.

FFT (fast Fourier transform) computes the Discrete Fourier transform of a sequence. This process converts a signal from its original domain (e.g. a time-domain graph shows how a signal changes over time) to a representation in the frequency domain (how a signal is distributed through many frequencies).[[3]](#footnote-3)

A screenshot of a graph

Description automatically generated

A discrete Fourier analysis of a sum of cosine waves at 10, 20, 30, 40, and 50 Hz  
(<https://en.wikipedia.org/wiki/Fast_Fourier_transform>)

There are many different types of FFT. I am going to use the Cooley-Tukey Algorithm, and to do that I am going to use a library called Accord.Math which has a function that takes an input of the data as an array of complex numbers (it uses complex numbers so that it can represent the phase of the wave as well as the amplitude) and a direction, which will be forward in this case.

….

|  |
| --- |
| Listener |
| + WaveIn waveIn;  - BufferedWaveProvider bwp;  - int RATE;  - int BUFFERSIZE;  + int DEVICENUMBER; |
| + Listener(int pBUFFERSIZE, int pRATE=44100, int pDeviceNumber=1)  - waveIn\_DataAvailable(object? Sender, WaveInEventArgs e)  + StartListening()  + string WhatNoteAmI(double frequency)  + List<string> ProcessData()  + double[] CalculatePCMValues(byte[] frames, int PointCount)  + double[] CalculateFFTValues(byte[] frames, int PointCount, double[] pcm)  + double[] DbScale(double[] fftReal)  + List<string> PullNotes(double[] fftRealDB, int PointCount) |



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# Development and testing

# Evaluation

1. https://visualstudio.microsoft.com/ [↑](#footnote-ref-1)
2. Adapted from information at https://en.wikipedia.org/wiki/Pulse-code\_modulation. [↑](#footnote-ref-2)
3. Adapted from information at https://en.wikipedia.org/wiki/Fast\_Fourier\_transform. [↑](#footnote-ref-3)